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Abstract

The cell-to-cell mapping (path integration) technique has been devised for MDOF non-linear and non-hysteretic systems subjected to random trains of mpulses driven by an ordinary renewal point process with gamma-distributed integer parameter interarrival times (an Erlang process). Since the renewal point process has not independent increments the state vector of the system, consisting of the generalized displacements and velocities, is not a Markov process. Initially it is shown how the indicated system can be converted to an equivalent Poisson driven system at the expense of introducing additional discrete-valued state variables for which the stochastic equations are also formulated. Thereby the original non-Markov system is converted into an equivalent system which do possesses the Markov property. Next the integro-differential master equation governing the probability density of the response of the equivalent Markov system is solved by means of path integration technique. The idea is to consider the transition times of the resulting Markov chain, for which at most one impulse is likely to arrive. The convection and lumping of the probability mass is determined by a method devised previously by Köylüoğlu and the authors for Poisson driven systems, which has been suitably modified with due account to the artificial, auxiliary, state variables introduced in the description. The method has been applied to the Duffing oscillator, and the results for the stationary probability density function have been compared with those obtained from extensive Monte Carlo simulations.

Key words: non-linear systems, random vibration, random impulses, renewal point process, path integration, cell-to-cell mapping

1 Introduction

In some problems of engineering the excitation, or its discontinuous component, may be characterized as a train of pulses arriving at random times.

In the simplest approach such excitations may be regarded as Poisson-distributed trains of impulses, in which case the impulses occurrence times are generated by the Poisson counting process. The problem of non-linear dynamical systems subjected to Poisson-distributed random impulses was dealt with by Roberts (1972), who proposed a perturbation solution to the

generalized Fokker-Planck-Kolmogorov equation governing the response probability density in that case. Tylikowski and Marowski (1986) applied the equivalent linearization technique to such a problem, using the exact moments of the response of the linearized system.

For non-linear systems with cubic non-linearity subjected to Poisson-distributed random impulses the moment equations technique combined with an ordinary cumulant-neglect closure technique (Iwankiewicz and Nielsen 1992), or with a modified cumulant-neglect closure technique (Iwankiewicz et al. 1990) has been developed. In the paper (Iwankiewicz and Nielsen 1994) the technique of moment equations has been extended from a Poisson to a renewal impulse process, with gamma distributed, with $k = 2$, interarrival times of impulses (an Erlang process with $k = 2$). The renewal impulse process has been recast, with the help of a random telegraph process, as a transformation of a stationary Poisson counting process. This allowed to effectively average the differential rule and to derive the equations for moments. In the paper (Nielsen et al. 1995) the moment equations technique has been further extended, using a different technique, to the whole class of renewal driving processes, with gamma distributed,with an arbitrary integer parameter k, interarrival times. Suitable (sine and cosine) transformations of the Poisson counting process, which only assume discrete values, are introduced as extra state variables. Such an approach allows to recast renewal driven system into an equivalent augmented dynamic system driven by a compound Poisson process. At the expense of augmenting the state vector by the additional, discrete-valued, variables the original non-Markov problem has thus been converted to a Markov one.

The main disadvantage, or shortcoming, of the moments equations technique for Poisson driven problems is that it fails for the trains of impulses with low mean arrival rates (Iwankiewicz and Nielsen 1992, 1994), (Iwankiewicz et al. 1990), (Nielsen et al. 1995).

Instead, the numerical solution of the generalized Fokker-Planck-Kolmogorov equation governing the probability of the response may be attempted. For example Cai and Lin (1992) developed a perturbation technique based on known exact solutions for the case of the Gaussian impulsive noise. Another approach is to solve the integro-differential analogue of the Kolmogorov backward equation with the help of a Petrov-Galerkin method (Köylüoğlu et al. 1994a). However the latter approach is also suitable only for relatively dense trains of impulses.

Cell-to-cell mapping technique for the problems of Poisson driven systems has been devised by Köylüoğlu et al. (1994b, 1995). The idea due to Sun and Hsu (1988, 1990) has been used to discretize the state space and hence to convert the continuous Markov process to a Markov chain problem, and the transition probability matrix has been constructed based on the assumption that for sufficiently small transition time interval the approximate transition probability density may be eavaluated by neglecting the probability of occurrence of more than one impulse in this time interval. Next the suitable scheme is devised of convecting and diffusing the probability mass in the state space (Köylüoğlu et al. 1995). An alternative, more effective version of the cell-to-cell mapping technique for the problems of Poisson driven systems, was developed Köylüoğlu et al. (1994b), which consists in expanding the convection term as a Taylor series in the random impulse magnitude. The random waiting time disappears from the equations at the expense of increasing the number of initial value problems for all Taylor-expansion coefficients,

which must be solved numerically.

In the present paper the renewal driven system is first converted to a Poisson driven one, using the same approach as in (Nielsen et al. 1995). Next, the cell-to-cell mapping technique is applied to solve the master equation of the resulting Markov problem. The approach is basically the same as in (Köylüoğlu et al. 1994b). However, significant modifications need to be introduced with due account to the auxiliary state variables. The numerical example concerns the Duffing oscillator subject to renewal impulses with gamma distributed, with $k = 2$, interarrival times.

2 Governing stochastic integro-differential equations for a class of renewal driven non-Markov response problems.

Consider a general multi-degree-of-freedom non-linear dynamical system under a random train of impulses driven by a renewal point process. The state vector of the system, $\mathbf{Z}_1(t)$, consisting of the generalized displacements and velocities, is governed by the set of equations of motion

$$\begin{aligned} \frac{d}{dt}\mathbf{Z}_1(t) &= \mathbf{c}_1\left(\mathbf{Z}_1(t),t\right) + \mathbf{d}_1(\mathbf{Z}_1(t),t)\sum_{i=1}^{R(t)} P_{i,R}\delta\left(t-t_{i,R}\right), \quad t > t_0, \\ \mathbf{Z}_1(t_0) &= \mathbf{z}_{1,0}, \end{aligned} \tag{1}$$

where $\mathbf{c}_1\left(\mathbf{Z}_1(t),t\right)$ is the drift vector, $\mathbf{d}_1(\mathbf{Z}_1(t),t)$ is an analogue of the diffusion vector in white noise driven problems and $\mathbf{z}_{1,0}$ denotes the initial values of the structural variables at the initial time t_0.

The occurrence times $t_{i,R}$ of Dirac delta impulses are distributed according to an ordinary renewal counting process $\{R(t), t \in [0,\infty[\}, \Pr\{R(0) = 0\} = 1$. The mark variables $P_{i,R}$ are assumed to be independent, identically distributed, random variables, independent of the occurrence times $t_{i,R}$ and having the distribution as a random variable P. It is obvious that since the renewal process is not a process with independent increments, the state vector $\mathbf{Z}_1(t)$ governed by the equation (1) is not a Markov process.

Let us confine the considerations to a class of Erlang renewal processes, i.e. the ones for which the interarrival times $X_n = t_n - t_{n-1}$ are independent, gamma distributed random variables, i.e. $X_n \sim \mathrm{G}(k-1,\nu)$, with the probability density function given by

$$f_{X_n}(t) = \nu^k t^{k-1}\exp(-\nu t)/(k-1)!, \quad t > 0, \tag{2}$$

where $k = 1,2,3,\ldots$. The case $k = 1$ corresponds to a negative exponential density function, hence it is the case of a Poisson process. Since the gamma distributed random variable with parameters $(k-1,\nu)$ has the same distribution as the sum of k independent, negative exponential

distributed random variables, with parameter ν, the events driven by an Erlang process with parameter k can be regarded as every kth Poisson events, taken out of a stationary Poisson process with the mean arrival rate ν, cf. e.g. (Osaki 1992).

The idea is to recast the renewal-driven impulse process, or the excitation term of the equation (1) in such a way as to obtain a non-zero impulse magnitude for every $k, 2k, 3k, \ldots$ i.e. every kth Poisson event and zero magnitudes for all other Poisson events. The Poisson counting process $\{N(t), t \in [0, \infty[\}$ is defined as the number of events in the time interval $[0, t[$, hence the additional assumption is made: $\Pr\{N(0) = 0\} = 1$.

Hence the governing stochastic equations should be converted to the form of

$$\frac{d}{dt}\mathbf{Z}_1(t) = \mathbf{c}_1\left(\mathbf{Z}_1(t), t\right) + \mathbf{d}_1(\mathbf{Z}_1(t), t) \sum_{i=1}^{N(t)} \rho(N\left(t_i\right)) P_i \delta\left(t - t_i\right), \tag{3}$$

where $\rho\left(N(t_i)\right)$ is the required transformation of the Poisson counting process $N(t_i)$, such that $\rho\left(N(t_i)\right) = 1$ for every kth Poisson event and $\rho\left(N(t_i)\right) = 0$ for all other Poisson events, $N(t_i)$ being the number of past Poisson events, not including the one which occurs at the time t_i.

The stochastic integro-differential equations governing the system state vector can then be written as

$$d\mathbf{Z}_1(t) = \mathbf{c}_1\left(\mathbf{Z}_1(t), t\right) dt + \mathbf{d}_1(\mathbf{Z}_1(t), t) \rho\left(N(t)\right) \int_{\mathcal{P}} pM(dt, t, dp, p), \tag{4}$$

where $M(dt, t, dp, p)$ is a Poisson random measure (Snyder 1975), which gives the random number of impulses in the time interval $[t, t + dt[$ with the random magnitudes from the interval $[p, p + dp[$. The expectation of this measure is, in the case of the stationary Poisson process

$$E[M(dt, t, dp, p)] = \nu f_P(p) dt dp, \tag{5}$$

where ν is the constant mean arrival rate of impulses and $f_P(p)$ is the probability density function of the random impulse magnitude P.

3 Converting the non-Markov problem to a Markov one by suitable recasting of the excitation process.

The transformation satisfying the required property is found to be

$$\rho(N(t)) = \frac{1}{k} \sum_{j=0}^{k-1} \exp\left(i2\pi \frac{j(N(t)+1)}{k}\right) = \frac{1}{k} \sum_{j=0}^{k-1} U_j, \tag{6}$$

$$U_j = \exp\left(i2\pi \frac{j(N(t)+1)}{k}\right). \tag{7}$$

For $N(t) = 0, 1, 2, \ldots, k-1$, U_j is the Finite Fourier Transform of the k-dimensional sequence $\{0, 0, \ldots, 0, 1\}$. From the periodicity properties it follows that

$$\rho(N(t)) = \begin{cases} 1 & , \text{ for } N(t) = k-1,\ 2k-1,\ 3k-1,\ldots \\ 0 & , \text{ else} \end{cases} \tag{8}$$

which means that $\rho(N(t)) = 1$ as every kth Poissonian impulse arrives.

Seeing that $U_j = U_{k-j}^*$, where * denotes the complex conjugate, the right-hand side of (6) can be evaluated as

$$\begin{aligned} \rho(N(t)) &= \begin{cases} \frac{1}{k}\left(1 + U_1 + \ldots + U_{k_0-1} + U_{k_0-1}^* + \ldots + U_1^*\right) & , \text{ for } k \text{ odd} \\ \frac{1}{k}\left(1 + U_1 + \ldots + U_{k_0-1} + U_{k_0} + U_{k_0-1}^* + \ldots + U_1^*\right) & , \text{ for } k \text{ even} \end{cases} \\ &= \begin{cases} \frac{1}{k}\left(1 + 2\sum_{j=1}^{k_0-1} C_j\right) & , \text{ for } k \text{ odd} \\ \frac{1}{k}\left(1 + 2\sum_{j=1}^{k_0-1} C_j + C_{k_0}\right) & , \text{ for } k \text{ even} \end{cases} \end{aligned} \tag{9}$$

where

$$k_0 = \left[\frac{k+1}{2}\right] \tag{10}$$

$[\ldots]$ being the integer part and

$$C_j(t) = \Re(U_j) = \cos\left(2\pi\frac{j(N(t)+1)}{k}\right), \quad j = 1,2,\ldots,k_0-1, \tag{11}$$

$$S_j(t) = \Im(U_j) = \sin\left(2\pi\frac{j(N(t)+1)}{k}\right), \quad j = 1,2,\ldots,k_0-1, \tag{12}$$

$$C_{k_0}(t) = \exp\left(i\pi(N(t)+1)\right) = (-1)^{N(t)+1} = \cos\Big(\pi(N(t)+1)\Big), \quad \text{for } k \text{ even}. \tag{13}$$

It should be noticed that these transformations of a Poisson counting process $N(t)$ are all discrete valued random processes. For example in the case $k = 2$, $C_{k_0}(t) = C_1(t) = \cos\Big(\pi(N(t)+1)\Big) = (-1)^{N(t)+1}$ only takes values $C_1(t) = 1$ and $C_1(t) = -1$.

These transformations will be regarded as additional, auxiliary, state variables. The stochastic equations for these variables are obtained from

$$\begin{aligned} dU_j(t) &= U_j(t+dt) - U_j(t) = \exp\left(i2\pi\frac{j(N(t+dt)+1)}{k}\right) - \exp\left(i2\pi\frac{j(N(t)+1)}{k}\right) \\ &= \exp\left(i2\pi\frac{j(N(t)+dN(t)+1)}{k}\right) - \exp\left(i2\pi\frac{j(N(t)+1)}{k}\right) \\ &= U_j(t)\left(\exp\left(i2\pi\frac{j}{k}dN(t)\right) - 1\right). \end{aligned} \tag{14}$$

The increment $dN(t)$ of the regular counting process can be either 0 or 1, with non-vanishing probabilities. Equation (14) can then be written as

$$dU_j(t) = U_j(t)\left(\exp\left(i2\pi\frac{j}{k}\right) - 1\right)dN(t), \quad j = 1, 2, \ldots, k_0. \tag{15}$$

The equivalence of (14) and (15) follows from the fact that the right-hand sides give the same result for both $dN(t) = 0$ and $dN(t) = 1$.

Specifically, the equations for the real and imaginary parts become

$$dC_j = \left(C_j\left(\cos\left(2\pi\frac{j}{k}\right) - 1\right) - S_j \sin\left(2\pi\frac{j}{k}\right)\right)dN(t), \quad j = 1, 2, \ldots, k_0 - 1, \tag{16}$$

$$dS_j = \left(C_j \sin\left(2\pi\frac{j}{k}\right) + S_j\left(\cos\left(2\pi\frac{j}{k}\right) - 1\right)\right)dN(t), \quad j = 1, 2, \ldots, k_0 - 1, \tag{17}$$

$$dC_{k_0} = -2C_{k_0}dN(t), \quad \text{for} \quad k \quad \text{even}. \tag{18}$$

It is seen that the additional state variables have been introduced:C_j, S_j and for k even also C_{k_0}. The state vector augmented by these new variables is governed by the stochastic equations

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t), t\big)dt + \int_{\mathcal{P}} \mathbf{d}\big(\mathbf{Z}(t), t, p\big)M(dt, t, dp, p), \tag{19}$$

where

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{bmatrix}, \quad \mathbf{c}(\mathbf{Z}(t), t) = \begin{bmatrix} \mathbf{c}_1(\mathbf{Z}_1(t), t) \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{d}(\mathbf{Z}(t), t, p) = \begin{bmatrix} \rho(N(t))\mathbf{d}_1(\mathbf{Z}_1(t), t)p \\ \mathbf{d}_2(\mathbf{Z}_2(t)) \end{bmatrix}, \tag{20}$$

where for k even

$$\mathbf{Z}_2(t) = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ C_{k_0-1} \\ S_{k_0-1} \\ C_{k_0} \end{bmatrix}, \quad \mathbf{d}_2(\mathbf{Z}_2(t)) = \begin{bmatrix} C_1\left(\cos\left(2\pi\frac{1}{k}\right) - 1\right) - S_1 \sin\left(2\pi\frac{1}{k}\right) \\ C_1 \sin\left(2\pi\frac{1}{k}\right) + S_1\left(\cos\left(2\pi\frac{1}{k}\right) - 1\right) \\ C_2\left(\cos\left(2\pi\frac{2}{k}\right) - 1\right) + S_2 \sin\left(2\pi\frac{2}{k}\right) \\ C_2 \sin\left(2\pi\frac{2}{k}\right) + S_2\left(\cos\left(2\pi\frac{2}{k}\right) - 1\right) \\ \cdot \\ \cdot \\ \cdot \\ C_{k_0-1}\left(\cos\left(2\pi\frac{k_0-1}{k}\right) - 1\right) - S_{k_0-1} \sin\left(2\pi\frac{k_0-1}{k}\right) \\ C_{k_0-1} \sin\left(2\pi\frac{k_0-1}{k}\right) + S_{k_0-1}\left(\cos\left(2\pi\frac{k_0-1}{k}\right) - 1\right) \\ -2C_{k_0} \end{bmatrix}, \tag{21}$$

and for k odd

$$\mathbf{Z}_2(t) = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ C_{k_0-1} \\ S_{k_0-1} \end{bmatrix}, \quad \mathbf{d}_2(\mathbf{Z}_2(t)) = \begin{bmatrix} C_1\Big(\cos(2\pi\frac{1}{k}) - 1\Big) - S_1 \sin(2\pi\frac{1}{k}) \\ C_1 \sin(2\pi\frac{1}{k}) + S_1\Big(\cos(2\pi\frac{1}{k}) - 1\Big) \\ C_2\Big(\cos(2\pi\frac{2}{k}) - 1\Big) + S_2 \sin(2\pi\frac{2}{k}) \\ C_2 \sin(2\pi\frac{2}{k}) + S_2\Big(\cos(2\pi\frac{2}{k}) - 1\Big) \\ \cdot \\ \cdot \\ \cdot \\ C_{k_0-1}\Big(\cos(2\pi\frac{k_0-1}{k}) - 1\Big) - S_{k_0-1} \sin(2\pi\frac{k_0-1}{k}) \\ C_{k_0-1} \sin(2\pi\frac{k_0-1}{k}) + S_{k_0-1}\Big(\cos(2\pi\frac{k_0-1}{k}) - 1\Big) \end{bmatrix}. \tag{22}$$

The state vector $\mathbf{Z}(t)$, augmented by additional, auxiliary state variables, as governed by equation (19) is driven by a Poisson process, and hence it is a Markov vector process. The structural state variables (generalized displacements and velocities) $\mathbf{Z}_1(t)$ are continuous, whereas the auxiliary state variables $\mathbf{Z}_2(t)$ are all discrete.

The transition probability density $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_0, t_0)$ of the non-diffusive, Poisson driven Markov vector process $\mathbf{Z}(t)$ is governed by the following integro-differential master equation obtained from the general forward integro-differential Chapman-Kolmogorov equation [see (Snyder 1975) and (Gardiner 1985)]

$$\begin{aligned} &\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_0, t_0) = -\sum_i \frac{\partial}{\partial z_i}[c_i(\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_0, t_0)] \\ &+\nu \int_{\mathcal{P}} \Big[q_{\{\mathbf{Z}\}}(\mathbf{a}(\mathbf{z}, p, t), t|\mathbf{z}_0, t_0) \frac{1}{|\mathcal{J}|} - q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_0, t_0)\Big] f_P(p) dp, \end{aligned} \tag{23}$$

where

$$\mathcal{J} = \det\left(\mathbf{I} + \frac{\partial \mathbf{d}(\mathbf{a}(\mathbf{z}, p, t), t)}{\partial \mathbf{y}^T}\right), \tag{24}$$

and $\mathbf{a}(\mathbf{z}, p, t)$ defined by

$$\mathbf{y} = \mathbf{a}(\mathbf{z}, p, t) \tag{25}$$

is the inverse transformation of

$$\mathbf{z} = \mathbf{y} + \mathbf{d}(\mathbf{y}, p, t) \tag{26}$$

and $\frac{\partial \mathbf{d}}{\partial \mathbf{y}^T}$ is the gradient of $\mathbf{d}(\mathbf{y}, p, t)$ with respect to $\mathbf{y}$.

The joint probability density $f_{\{\mathbf{Z}\}}(\mathbf{z}, t)$ of the state vector $\mathbf{Z}(t)$ is governed by the same equation, (Iwankiewicz and Nielsen 1995), cf. also (Renger 1979)

$$\frac{\partial}{\partial t} f_{\{\mathbf{Z}\}}(\mathbf{z}, t) = -\sum_i \frac{\partial}{\partial z_i}[c_i(\mathbf{z}, t) f_{\{\mathbf{Z}\}}(\mathbf{z}, t] + \nu \int_{\mathcal{P}} \Big[f_{\{\mathbf{Z}\}}(\mathbf{a}(\mathbf{z}, p, t), t) \frac{1}{|\mathcal{J}|} - f_{\{\mathbf{Z}\}}(\mathbf{z}, t)\Big] f_P(p) dp. \tag{27}$$

4 Cell-to-cell mapping technique for the Poisson-driven problem with the state dependent diffusion vector

Let $q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0)$ be the transition probability density function of the state vector from the state $\mathbf{Z}(t_0) = \mathbf{z}_0$ on condition of exact n impulse arrivals in the time interval $[t_0, t_1[$, and let $P_{\{N\}}(n, t_1|N(t_0) = 0)$ denote the probability function of exactly n arrivals in this time interval. Using the total probability theorem the unconditional transition probability density function can then be written as

$$q_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0) = \sum_{n=0}^{\infty} P_{\{N\}}(n, t_1|N(t_0) = 0) q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0). \tag{28}$$

The conditional probability function for the stationary Poisson process is given as

$$P_{\{N\}}(n, t_1|N(t_0) = 0) = \frac{(\nu(t_1 - t_0))^n}{n!} \exp\left(-\nu(t_1 - t_0)\right). \tag{29}$$

$q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0)$ describes the purely deterministic drift of the system from the state $\mathbf{Z}(t_0) = \mathbf{z}_0$, since the states are conditioned on no impulse arrival. The structural state variables $\mathbf{Z}_1(t_1)$ are changed only because of the eigenvibration of the system. The auxiliary state variables $\mathbf{Z}_2(t_1)$ remain constant at their initial values $\mathbf{Z}_2(t_0) = \mathbf{z}_{2,0}$, since these variables are only changed at Poisson impulse arrivals. Then $q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0)$ is given as

$$q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0) = \delta\Big(z_1 - e_1(t_1|\mathbf{z}_0, t_0)\Big) \cdots \delta\Big(z_n - e_n(t_1|\mathbf{z}_0, t_0)\Big), \tag{30}$$

where $\mathbf{e}(t_1|\mathbf{z}_0, t_0)$ with the components $e_i(t_1|\mathbf{z}_0, t_0)$ denotes the deterministic drift motion of the system from the initial state $\mathbf{z}_0$ at the time t_0. The vector $\mathbf{e}(t|\mathbf{z}_0, t_0)$ consists of the subvectors $\mathbf{e}_1(t|\mathbf{z}_{1,0}, t_0)$ and $\mathbf{e}_2(t|\mathbf{z}_{2,0}, t_0) \equiv \mathbf{z}_{2,0}$ corresponding to structural and auxiliary state variables, respectively. The vector $\mathbf{e}_1(t|\mathbf{z}_{1,0}, t_0)$ is the solution of the initial value problem originating from (1)

$$\frac{\partial}{\partial t}\mathbf{e}_1(t|\mathbf{z}_{1,0}, t_0) = \mathbf{c}_1\Big(\mathbf{e}_1(t|\mathbf{z}_{1,0}, t_0), t\Big), \quad t > t_0, \quad \mathbf{e}_1(t_0|\mathbf{z}_{1,0}, t_0) = \mathbf{z}_{1,0}. \tag{31}$$

The vector $\mathbf{e}_1(t|\mathbf{z}_{1,0}, t_0)$ may be obtained analytically for linear systems depending on a fundamental basis of solutions can be found. This is the case of linear vibratory systems, where modal decoupling can be used. For other linear systems and for non-linear systems the numerical integration of (31) is necessary.

The remaining conditional transition probability density functions, $q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0)$, $n \geq 1$, are all continuous functions without delta spikes, and of the same order of magnitude. Since $P_{\{N\}}(n, t_1|N(t_0) = 0) = \mathrm{O}\Big((\nu\Delta t)^n\Big), \Delta t = t_1 - t_0$, it follows, that (28) can be written as

$$\begin{aligned} q_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0) &= P_0(t_1|t_0) q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0) + \\ &\Big(1 - P_0(t_1|t_0)\Big) q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0) + \mathrm{O}\Big((\nu\Delta t)^n\Big), \end{aligned} \tag{32}$$

$$P_0(t_1|t_0) = P_{\{N\}}(N(t_1) = 0|N(t_0) = 0) = \exp\left(-\nu(t_1 - t_0)\right). \tag{33}$$

The asymptotic relationship (32) forms the basis for cell-to-cell mapping methods for pulse excited systems (Köylüoğlu et al. 1994b, 1995). The specific formulation ensures that upon chopping the remainder (32), the quality of a genuine (actual) probability density function is preserved, i.e. the integral of the function over the sample space is exactly equal to one, for any choice of the transition time interval $\Delta t = t_1 - t_0$. Further, it is important to notice that the remainder depends on the magnitude of the product $\nu\Delta t$, rather than of the interval length $\Delta t = t_1 - t_0$ itself. Truncation is permitted if

$$\nu\Delta t << 1. \tag{34}$$

For any finite transition interval (34) is more easily fulfilled for sparse pulse trains ($\nu << 1$) than for the dense pulse trains, hence in contrast to the moment equations method for Poisson and renewal driven impulses (Iwankiewicz and Nielsen 1992, 1994), (Iwankiewicz et al. 1990), (Nielsen et al. 1995) and the Petrov-Galerkin finite element formulation (Köylüoğlu et al. 1994a), which both work well for rather dense pulse trains and run into numerical instability for sparse pulses, the cell-to-cell mapping technique appears to be especially effective for sparse pulses and less effective for dense ones (Köylüoğlu et al. 1994b, 1995). On the other hand the coarse size of the mesh sets lower bounds for the admissibility of the transition time interval, Δt. This is so, because the distribution of the convected probability masses to the adjacent nodes in the mesh cannot be done accurately, if $\Delta t = t_1 - t_0$ is too small.

At the determination of $q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t_1|\mathbf{z}_0, t_0)$ the state is conditioned on exactly one impulse arrival in $[t_0, t_1[$. The arrival time (waiting time) T, of this impulse is uniformly distributed in the interval $[t_0, t_1[$, and hence it has the probability density function, (Osaki 1992)

$$f_T(\tau) = \frac{\tau}{t_1 - t_0}, \quad t_0 < \tau < t_1. \tag{35}$$

Assume that the impulse arrives at the time $T = \tau$ and has the strength $P = p$. Up to the time τ the system has been performing eigenvibrations in the structural state variables from the initial state $\mathbf{z}_{1,0}$ at the time t_0 and the auxiliary state variables are kept constant at their initial values $\mathbf{z}_{2,0}$, hence $\rho(N(\tau)) = \rho(N(t_0))$. Then the state at the time τ^- is given by $\mathbf{Z}_1(\tau^-) = \mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0)$ and $\mathbf{Z}_2(\tau^-) = \mathbf{z}_{2,0}$. At the time τ a discontinuous change of the state of magnitude given by $\Delta\mathbf{Z}_1(\tau) = \rho(N(t_0))\mathbf{d}_1(\mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0), \tau)p$ and $\Delta\mathbf{Z}_2(\tau) = \mathbf{d}_2(\mathbf{z}_{2,0})$ takes place, so the state at the time τ^+ becomes $\mathbf{Z}_1(\tau^+) = \mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0) + \rho(N(t_0))\mathbf{d}_1(\mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0), \tau)p$ and $\mathbf{Z}_2(\tau^+) = \mathbf{z}_{2,0} + \mathbf{d}_2(\mathbf{z}_{2,0})$. This implies a discontinuity in the generalized velocities and in the auxiliary variables, whereas the generalized displacements remain inaffected. Succeedingly, during the time interval $]\tau, t_1]$ the auxiliary variables are kept constant at the value $\mathbf{z}_{2,0}+\mathbf{d}_2(\mathbf{z}_{2,0})$ whereas the system continues performing eigenvibrations in the structural state variables with the initial values $\mathbf{Z}_1(\tau^+)$, so the state at the time t_1 is

$$Z_{1,i}(t_1) = e_{1,i}\Big(t_1|\mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0) + \rho(N(t_0))\mathbf{d}_1(\mathbf{e}_1(\tau|\mathbf{z}_{1,0}, t_0), \tau)p, \tau\Big). \tag{36}$$

Obviously the following identity can be formulated for the structural components of the deterministic drift, for any $\tau \in [t_0, t_1[$

$$\mathbf{e}_1\Big(t_1\Big|\mathbf{e}_1(\tau|\mathbf{z}_{1,0},t_0),\tau\Big) = \mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big). \tag{37}$$

The left- and right-hand side of (37) just state that the oscillator arrives at the same position at the time t, if it starts on the very same trajectory at the place $\mathbf{e}_1(\tau|\mathbf{z}_{1,0},t_0)$ at the time τ or at the place $\mathbf{e}_1(t_0|\mathbf{z}_{1,0},t_0) = \mathbf{z}_{1,0}$ at the time t_0. Assume that $\mathbf{d}_1(\mathbf{z}_1(t),t)$ can be assumed to be constant in the interval $[t_0,t_1[$ at its initial value $\mathbf{d}_1 = \mathbf{d}_1(\mathbf{z}_{1,0},t_0)$ with sufficient accuracy. Using (37) the following Taylor expansion in the diffusion vector $\rho(N(t_0))\mathbf{d}_1(\mathbf{z}_{1,0}(t_1),t_1))P$ dependent on the random time τ then prevails, (Köylüoğlu et al. 1994b,1995)

$$\begin{aligned}
&\mathbf{e}_1\Big(t_1\Big|\mathbf{e}_1(\tau|\mathbf{z}_{1,0},t_0)+\rho(N(t_0))\mathbf{d}_1P,\tau\Big) = \\
&\mathbf{e}_1\Big(t_1\Big|\mathbf{e}_1(\tau|\mathbf{z}_{1,0},t_0),\tau\Big)+\rho(N(t_0))P\sum_{l=1}^{2n}\frac{\partial\mathbf{e}_1\Big(t_1\Big|\mathbf{e}_1(\tau|\mathbf{z}_{1,0},t_0),\tau\Big)}{\partial z_{1,0,l}}d_{1,l}+\ldots = \\
&\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)+\rho(N(t_0))P\sum_{l=1}^{2n}\frac{\partial\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)}{\partial z_{1,0,l}}d_{1,l}+ \\
&\rho(N(t_0))\frac{P^2}{2}\sum_{l=1}^{2n}\sum_{m=1}^{2n}\frac{\partial^2\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)}{\partial z_{1,0,l}\partial z_{1,0,m}}d_{1,l}d_{1,m}+\ldots = \\
&\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)+\rho(N(t_0))\mathbf{e}_1^{(1)}\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)P+\rho(N(t_0))\mathbf{e}_1^{(2)}\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)P^2+\ldots
\end{aligned} \tag{38}$$

$$\mathbf{e}_1^{(1)}\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big) = \sum_{l=1}^{2n}\frac{\partial\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)}{\partial z_{1,0,l}}d_{1,l}(\mathbf{z}_{1,0},t_0) \tag{39}$$

$$\mathbf{e}_1^{(2)}\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big) = \frac{1}{2}\sum_{l=1}^{2n}\sum_{m=1}^{2n}\frac{\partial^2\mathbf{e}_1\Big(t_1\Big|\mathbf{z}_{1,0},t_0\Big)}{\partial z_{1,0,l}\partial z_{1,0,m}}d_{1,l}(\mathbf{z}_{1,0},t_0)d_{1,m}(\mathbf{z}_{1,0},t_0) \tag{40}$$

where it has been taken into account that $\rho(N(t_0)) = \rho^2(N(t_0)) = \rho^s(N(t_0))$, for any s, because $\rho(N(t_0))$ can only assume values 0 or 1. The state of auxiliary state variables at the time t_0 determines which of the two values is attained. Hence $\rho(N(t_0))$ is merely a function of $\mathbf{z}_{2,0}$.

Equation (38) is basically a Taylor expansion in the impulse magnitude P for the change of the structural state variables. At the same time the auxiliary state variables have changed from $\mathbf{z}_{2,0}$ to $\mathbf{z}_{2,0}+\mathbf{d}_2(\mathbf{z}_{2,0})$. It is remarkable that the random time τ at the left-hand side of (38) has totally disappeared at the right-hand side. Instead the unknown Taylor expansion vectors $\mathbf{e}_1, \mathbf{e}_1^{(1)}, \ldots$ appear. Differential equations for these quantities can be derived upon multiple partial differentiations of (31) with respect to $\mathbf{z}_{1,0}$ and subsequent contractions with respect to $\mathbf{d}_1(\mathbf{z}_{1,0},t_0)$ as follows

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{e}_1\left(t\middle|\mathbf{z}_{1,0},t_0\right)}{\partial \mathbf{z}_{1,0}^T}\right)=\frac{\partial \mathbf{c}_1\left(\mathbf{e}_1(t\middle|\mathbf{z}_{1,0},t_0),t\right)}{\partial \mathbf{z}_1^T}\frac{\partial \mathbf{e}_1\left(t\middle|\mathbf{z}_{1,0},t_0\right)}{\partial \mathbf{z}_{1,0}^T}\ ,\ \frac{\partial \mathbf{e}_1\left(t_0\middle|\mathbf{z}_{1,0},t_0\right)}{\partial \mathbf{z}_{1,0}^T}=\mathbf{I}\Rightarrow$$

$$\frac{\partial}{\partial t}\mathbf{e}_1^{(1)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)=\frac{\partial \mathbf{c}_1\left(\mathbf{e}_1(t\middle|\mathbf{z}_{1,0},t_0),t\right)}{\partial \mathbf{z}_1^T}\mathbf{e}_1^{(1)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)\ ,\ \mathbf{e}_1^{(1)}\left(t_0\middle|\mathbf{z}_{1,0},t_0\right)=\mathbf{d}_1(\mathbf{z}_{1,0},t_0),$$

$$\frac{\partial}{\partial t}\mathbf{e}_1^{(2)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)=\frac{\partial \mathbf{c}\left(\mathbf{e}_1(t\middle|\mathbf{z}_{1,0},t_0),t\right)}{\partial \mathbf{z}_1^T}\mathbf{e}_1^{(2)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)+ \tag{41}$$

$$\frac{1}{2}\frac{\partial^2 \mathbf{c}\left(\mathbf{e}_1(t\middle|\mathbf{z}_{1,0},t_0),t\right)}{\partial \mathbf{z}_1^T\partial \mathbf{z}_1^T}\mathbf{e}_1^{(1)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)\mathbf{e}_1^{(1)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)\ ,\ \mathbf{e}_1^{(2)}\left(t_0\middle|\mathbf{z}_{1,0},t_0\right)=\mathbf{0},$$

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where the partial differentiation with respect to the forward state $\mathbf{z}_1$ concerns the drift vector $\mathbf{c}_1(\mathbf{z}_1,t)$. Relationships (31) and (41) represent a coupled system of $2n+2n+2n+\ldots$ non-linear 1st order differential equations for the determination of $\mathbf{e}_1\left(t\middle|\mathbf{z}_{1,0},t_0\right)$, $\mathbf{e}_1^{(1)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)$, $\mathbf{e}_1^{(2)}\left(t\middle|\mathbf{z}_{1,0},t_0\right),\ldots$.

For a linear system the drift vector is a linear function of the state vector, hence $\frac{\partial^2}{\partial \mathbf{z}_1^T\partial \mathbf{z}_1^T}\mathbf{c}_1(\mathbf{z}_1,t)\equiv\mathbf{0}$. Due to the homogeneous initial values $\mathbf{e}_1^{(2)}\left(t_0\middle|\mathbf{z}_{1,0},t_0\right)=\mathbf{0}$, then (41) provide the solution $\mathbf{e}_1^{(2)}\left(t\middle|\mathbf{z}_{1,0},t_0\right)\equiv\mathbf{0}$. Generally, it can be shown, that $\mathbf{e}_1^{(N)}(t|\mathbf{z}_{1,0},t_0)\equiv\mathbf{0}$, $N\geq 1$, in this case. Consequently, the Taylor expansion (38) *becomes linear in P for linear systems.*

At small transition time intervals $\mathbf{c}_1(\mathbf{z}_1(t),t)$ can be approximated by a locally linear function. Since this approach works well even for rather large time steps in case of white noise driven systems, (Köylüoğlu et al. 1994b), (Sun and Hsu 1990), it will also do so in the present case of compound Poisson driven systems. Consequently, it can be concluded that one can chop all terms higher than the 1st power in P in the expansion (38) with pretty good accuracy independently of the magnitude of P.

In case of cell-to-cell mapping technique the state space of the structural state variables $\mathbf{Z}_1(t)$ is discretized using a rectangular mesh with a grid width $\Delta z_{1,i}$, ($\Delta z_{1,1}=\Delta y, \Delta z_{1,2}=\Delta \dot{y}$), as shown in fig.1. The auxiliary variables are already discrete as mentioned, hence their state space it is just a discrete set of specific values, which can be simultaneously assumed by different auxiliary state variables. Consequently the entire state space can be represented as a set of phase hyperplanes, one for each combination of the additional state variables. For example in the case $k=2$ the auxiliary state variable only assumes values -1 and $+1$ and the entire state space consists of two phase hyperplanes. In the case of $k=3$ there is a set of three pairs of values assumed independently by two auxiliary state variables, hence there will be 3 such hyperplanes.

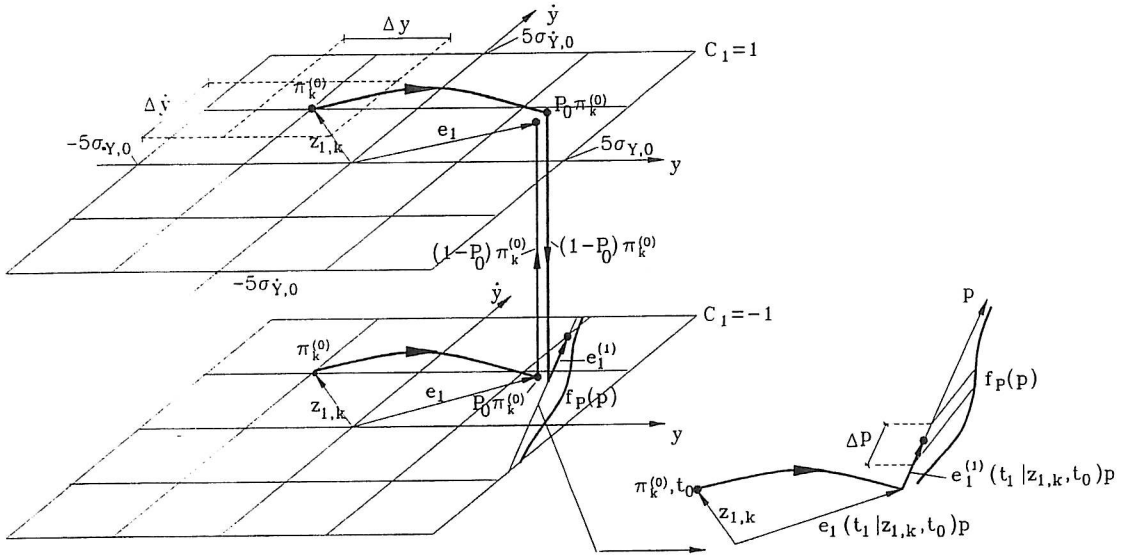

Fig. 1. Discretization of the state space for the case $k = 2$. Convection and lumping of the probability mass.

The convection and diffusion of the probability mass is illustrated in fig. 1 for the case $k = 2$. The particle is localized in the structural variables state space (hyperplane) at the nodal point $\mathbf{z}_{1,k}$, either at the hyperplane $C_1(t_0) = +1$ or $C_1(t_0) = -1$. Assume a probability mass $\pi_k^{(0)}$ is present at the node $\mathbf{z}_{1,k}$. The probability mass $P_0(t_1|t_0)\pi_k^{(0)}$ is then convected to the position $\mathbf{e}_1(t_1|\mathbf{z}_{1,k}, t_0)$ at the hyperplane $C_1(t_0)$ and redistributed to the neighbouring grid points according to their distances. The remaining probability mass $\big(1 - P_0(t_1|t_0)\big)\pi_k^{(0)}$ moves to the other hyperplane $-C_1(t_0)$ corresponding to a Poisson arrival. If $C_1(t_0) = -1$, then $\rho(N(t_0)) = 0$, and the probability mass $\big(1 - P_0(t_1|t_0)\big)\pi_k^{(0)}$ at the position $\mathbf{e}_1(t_1|\mathbf{z}_{1,k}, t_0)$ at the hyperplane $C_1(t_0) = +1$ is just redistributed to the adjacent nodes. If $C_1(t_0) = +1$, then $\rho(N(t_0)) = 1$, and the remaining probability mass $\big(1 - P_0(t_1|t_0)\big)\pi_k^{(0)}$ is distributed along the line $\mathbf{e}_1(t_1|\mathbf{z}_{1,k}, t_0) + \mathbf{e}_1^{(1)}(t_1|\mathbf{z}_{1,k}, t_0)p$, $p \in R$, at the hyperplane $C_1(t_0) = -1$ according to the probability density function $f_P(p)$ of the impulse magnitude. Impulses of the magnitude from the interval $[p, p + \Delta p[$ take place with the probability $f_P(p)\Delta p$. Hence, a probability mass of magnitude $\big(1 - P_0(t_1|t_0)\big)\pi_k^{(0)} f_P(p)\Delta p$ is displaced to $\mathbf{e}_1(t_1|\mathbf{z}_{1,k}, t_0) + \mathbf{e}_1^{(1)}(t_1|\mathbf{z}_{1,k}, t_0)p$, see fig. 1. Again this probability mass is redistributed to the adjacent grid points of the mesh.

If it is set $\pi_k^{(0)} = 1$, the resulting probability mass at the node j, denoted as Q_{jk}, is the element in the jth row and kth column of the transition probability matrix $\mathbf{Q}$. The probability

$\pi_j^{(i+1)}$ of being in the jth cell $\Delta \mathbf{z}_j$ at the time t_{i+1} is then given by

$$\pi_j^{(i+1)} = \sum_{k=1}^{N} Q_{jk} \pi_k^{(i)}, \quad j = 1, \ldots, N. \tag{42}$$

Let $\Delta \mathbf{z}_{1,k}$ be the volume of the mesh element centered at the structural state variables $\mathbf{z}_{1,k}$. The joint probability density function of the state variables then becomes

$$f_{\{\mathbf{Z}_1\}}(\mathbf{z}_{1,k}, t_i) \Delta \mathbf{z}_{1,k} = \sum_j \pi_j^{(i)}. \tag{43}$$

The sum of probabilities at the right-hand side of (43) is extended over all states in all hyperplanes with the same structural state variables $\mathbf{z}_{1,k}$.

The transition of states (42) can be represented by the matrix equation

$$\boldsymbol{\pi}^{(i+1)} = \mathbf{Q} \boldsymbol{\pi}^{(i)} = \mathbf{Q}^i \boldsymbol{\pi}^{(0)}, \tag{44}$$

where $\mathbf{Q}^i = \mathbf{Q} \cdots \mathbf{Q}$ ($\mathbf{Q}$ multiplied by itself i times), $\boldsymbol{\pi}^{(i)}$ is an N-dimensional vector of the state probabilities $\pi_k^{(i)}$ after ith transition, and $\boldsymbol{\pi}^{(0)}$ denotes the initial distribution at the time t_0. Since the considered stationary Markov chain is irreducible, positive recurrent and aperiodic, the stationary distribution $\boldsymbol{\pi}^{(\infty)}$ may be obtained after infinite many transitions as $i \to \infty$, [see e.g. (Osaki 1992)].

5 Example problem: Duffing oscillator driven by renewal impulses with gamma distributed, with $k = 2$, interarrival times

Consider a Duffing oscillator, with displacement $Y(t)$ and velocity $\dot{Y}(t)$, where $\mathbf{Z}_1(t)$, $\mathbf{c}_1(\mathbf{Z}_1(t), t)$ and $\mathbf{d}_1(\mathbf{Z}_1(t), t)$ of equation (1) are given by

$$\mathbf{Z}_1(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix}, \mathbf{c}_1(\mathbf{Z}_1(t), t) = \begin{bmatrix} \dot{Y}(t) \\ -2\zeta\omega_0 \dot{Y}(t) - \omega_0^2 Y(t) - \varepsilon \omega_0^2 Y^3(t) \end{bmatrix}, \mathbf{d}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{45}$$

where ζ is the damping ratio, ω_0 is the circular eigenfrequency of the corresponding linear oscillator and ε is the non-linearity parameter.

The data assumed for the Duffing oscillator is: $\omega_0 = 1s^{-1}$, $\zeta = 0.01$, $\varepsilon = 0.5$. With this value of the parameter ε the non-linearity should be regarded as quite strong, since the mean value and the variance of the response of the Duffing oscillator to the Poisson train of impulses are then substantially different from the statistics of the response of a linear oscillator, [cf. (Iwankiewicz and Nielsen 1994)].

Three different values of the parameter ν have been assumed: $\nu = 0.02$, $\nu = 0.2$ and $\nu = 2$. In order for the different cases of the renewal impulse proceses with different ν to be comparable, the data for the random variable P is assumed in such a way that $\nu E[P^2]$ is the same for all those cases. The reason for doing so is that the variance of the stationary response of a linear oscillator to a comparative Poissonian train of impulses, with the mean arrival rate $\frac{\nu}{2}$, as given by

$$\sigma^2_{Y,0} = \frac{\nu}{2}\frac{E[P^2]}{4\zeta\omega_0^3}. \tag{46}$$

is then the same. Moreover the data is assumed in such a way that $\sigma^2_{Y,0}$ as given by (46) has a unit value, hence $\nu E[P^2] = 0.08$ in all the considered cases. The impulses magnitudes have been assumed to be non-zero mean, Gaussian-distributed random variables, $P \sim N(\sigma_P, 0)$, with the density function

$$f_P(p) = \frac{1}{\sqrt{2\pi}\sigma_P}\exp\left(-\frac{p^2}{2\sigma_P^2}\right), \tag{47}$$

where $\sigma_P = \sqrt{0.08/\nu}$, because for a Gaussian distributed random variable $E[P^2] = \sigma_P^2$.

The stationary distributions have been evaluated after 60 transitions of eq. (44). The length Δt of a transition time interval has been determined by inspection for different values of ν. The best choice is $\Delta t = T_0$, $\Delta t = 0.2T_0$, $\Delta t = 0.05T_0$, for $\nu = 0.02$, $\nu = 0.2$ and $\nu = 2$, respectively, where $T_0 = 2\pi/\omega_0$ is the eigenperiod of a corresponding linear oscillator. A uniform 44×44 mesh has been applied with the limits $[-5\sigma_Y, 5\sigma_Y] \times [-5\sigma_{\dot{Y}}, 5\sigma_{\dot{Y}}]$. The stationary marginal probability densities of the displacement and velocity response of the Duffing oscillator evaluated for three different cases of ν are given in figures 2, 4 and 6. In order to show better the behaviour of the probability density tails the results are also given in the logarithmic scale. In addition the results of the investigations on the suitable choice of the transition time length Δt are given in figures 3, 5 and 7 for the displacement and velocity response probability densities represented in logarithmic scale.

The ergodic sampling technique has been used for the simulations. The interarrival times of renewal driven impulses with $k = 2$ are generated as the sum of two independent negative exponential distributed random variables. The response sample curve is next obtained by numerical integration, with the help of 4th order Runge-Kutta technique, of the homogeneous governing equation of motion (1) between the impulses arrival times, whereas at every second Poisson arrival the velocity is increased by a jump, which gives the updated initial condition for the next interarrival time interval. The response sample function of a length of $4000000T_0$ has been generated, sampled with an interval $\Delta t = T_0/40$, which is also the time step for the numerical integration. The same mesh was applied for the simulations as for the cell-to-cell mapping technique.

In table 1 the standard deviations σ_Y, $\sigma_{\dot{Y}}$ of the stationary displacement and velocity responses are shown, obtained by simulation and by the numerical technique (cell-to-cell mapping).

ν	$\Delta t/T_0$	σ_Y		$\sigma_{\dot{Y}}$	
		sim.	num.	sim.	num.
0.02	1.0	0.69938	0.70703	0.99768	1.00295
0.2	0.2	0.75587	0.79073	1.00138	1.06577
2.0	0.05	0.76174	0.64546	1.00151	0.80115

Table 1. The standard deviations σ_Y, $\sigma_{\dot{Y}}$ of the stationary displacement and velocity responses, for the considered values of ν.

It is seen in the table 1 that the predictions of the stationary response standard deviations, obtained from the numerical technique developed, are very good in the case of the lowest value of ν, i.e. $\nu = 0.02$, they are still quite good for $\nu = 0.2$, but they are no more satisfactory for $\nu = 2$.

Stationary response probability densities obtained with the help of the cell-to-cell mapping technique devised in the present paper are in a very good agreement with the results of simulations for a very low value of ν, i.e. $\nu = 0.02$, (hence for a very sparse train of impulses) as it is seen in the figure 2. For $\nu = 0.2$ this agreement is also good (figure 4), but it is becomes very bad for $\nu = 2$ (figure 6).

The optimal value of the length Δt of the transition time interval is determined in the case $\nu = 0.02$ as $\Delta t = T_0$ (figure 4), hence $\nu\Delta t = 0.02T_0$. In the case $\nu = 0.2$ it is $\Delta t = 0.2T_0$ (figure 5), giving $\nu\Delta t = 0.04T_0$, which is higher than in the previous case. Finally for $\nu = 2$, for the length of the transition time interval assumed as $\Delta t = 0.02T_0$ (which also gives $\nu\Delta t = 0.04$) the results are very inaccurate (figure 7). An improvement is attained by increasing Δt up to $\Delta t = 0.05T_0$, which gives $\nu\Delta t = 0.1T_0$. The value $\nu = 2$ seems to be the limit of the applicability of the cell-to-cell mapping technique developed, which performs very good for very sparse trains of impulses, i.e. for very low values of ν, cf. (Köylüoğlu et al. 1994b).

6 Concluding remarks

The cell-to-cell mapping technique has been devised for MDOF non-linear and non-hysteretic systems subjected to random trains of mpulses driven by an ordinary renewal point process with gamma-distributed integer parameter interarrival times. A crucial, initial step is converting the original renewal driven and hence non-Markov problem to a Poisson driven, Markov one. This is done by introducing additional discrete-valued state variables for which the stochastic equations are also formulated. The state vector of the system is thereby suitably augmented. The cell-to-cell mapping technique devised for the augmented Markov system is based on considering the transition time intervals, for which at most one impulse is likely to arrive. Next, the convection and lumping of the probability mass is determined with due account to the artificial, auxiliary, state variables introduced in the description. The method has been applied to the Duffing oscillator subjected to a renewal driven train of impulses with gamma distributed with $k = 2$, interarrival times. Three different values of parameter ν corresponding to low arrival rates

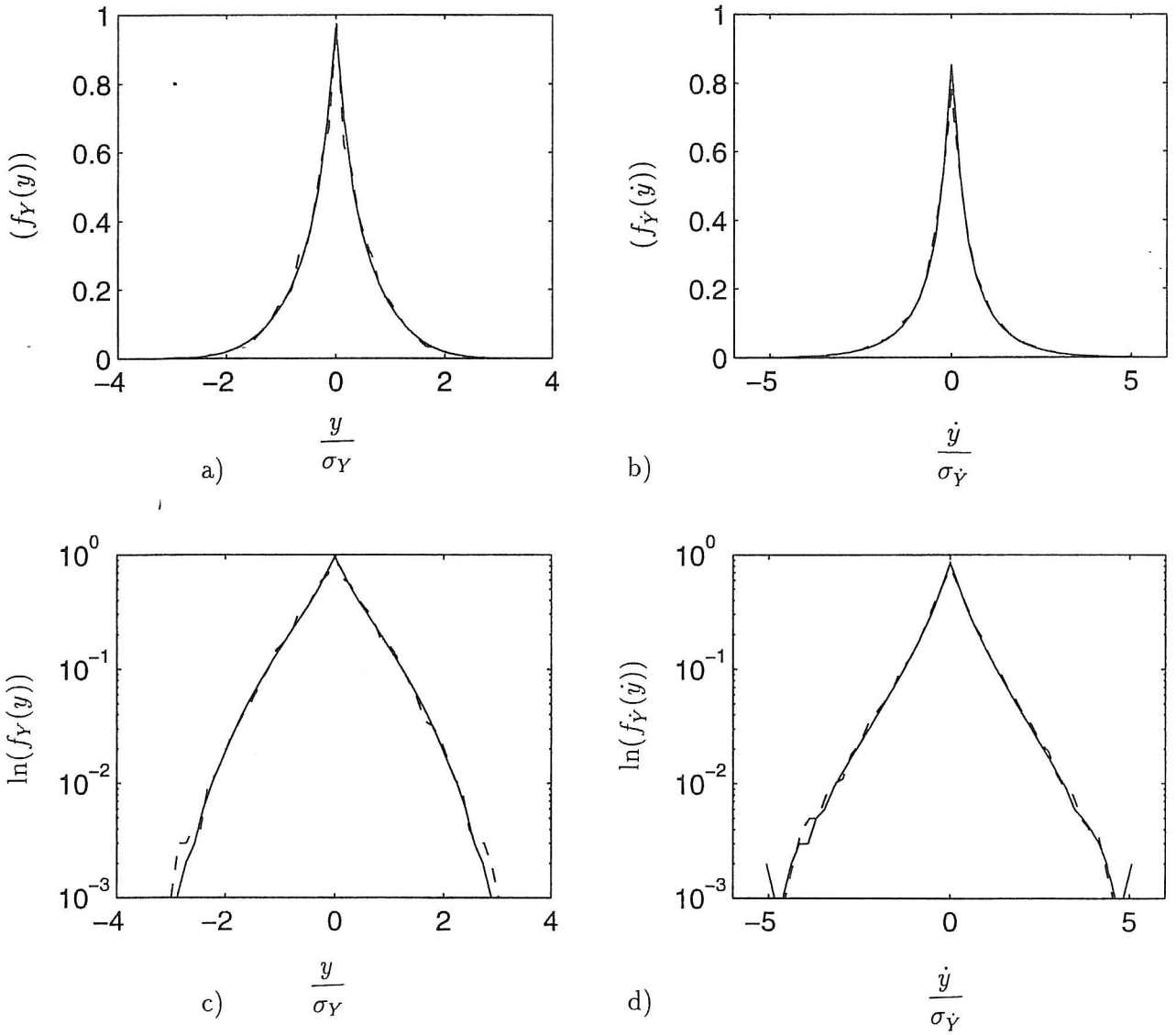

Fig. 2. Case $\nu = 0.02$. Stationary marginal probability densities of the response of Duffing oscillator driven by the renewal impulse process. a) Stationary probability density function $f_Y(y)$ of the displacement response. b) Stationary probability density function $f_{\dot{Y}}(\dot{y})$ of the velocity response. c) Stationary probability density function of the displacement response, logarithmic scale. d) Stationary probability density function of the velocity response, logarithmic scale. (- -): cell-to-cell mapping results, (———): simulation results, $\Delta t = T_0$.

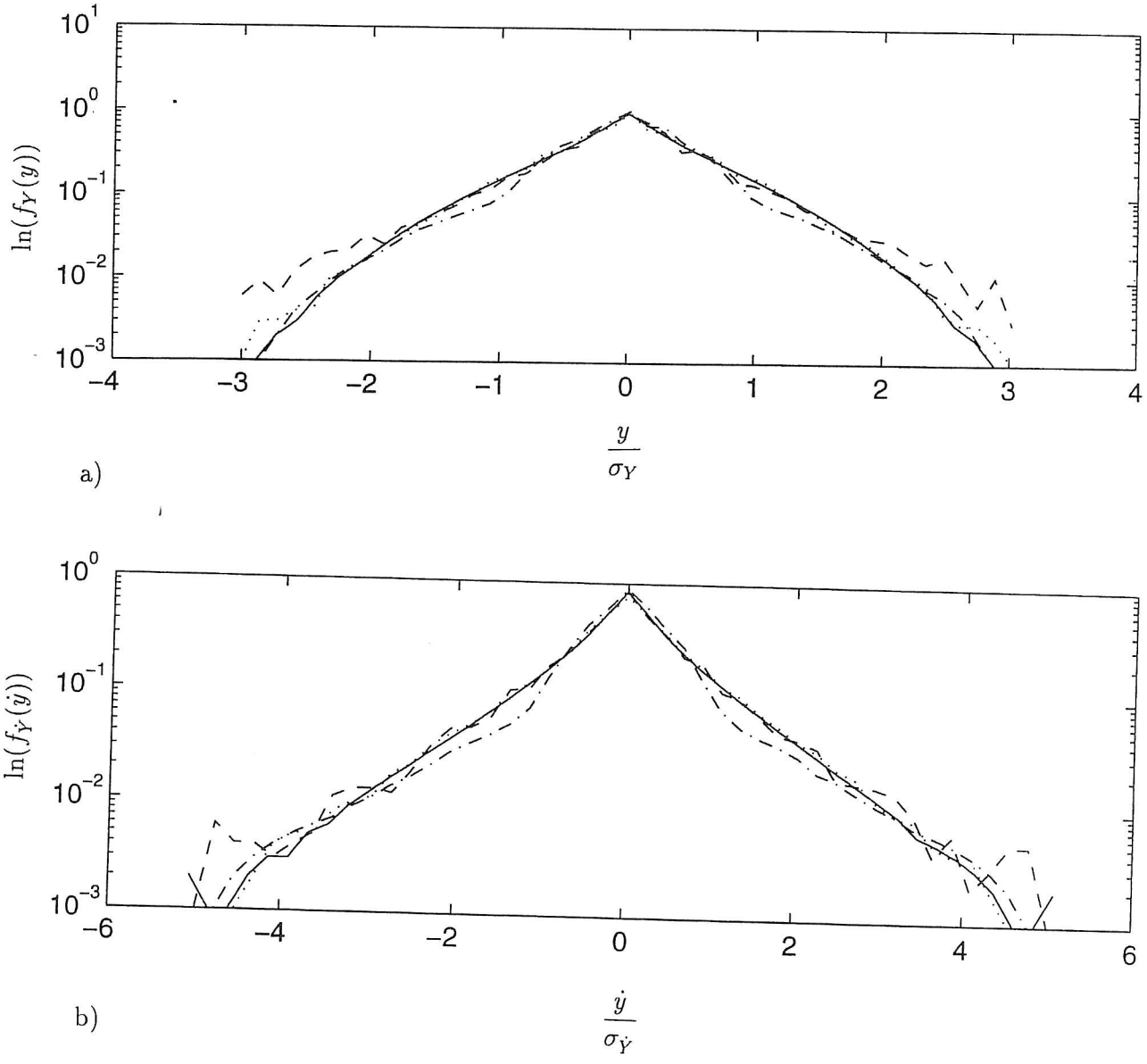

Fig. 3. Case $\nu = 0.02$. Results obtained for different choices of the length Δt of a transition time interval. a) Stationary probability density function of the displacement response, in logarithmic scale. b) Stationary probability density function of the velocity response, in logarithmic scale. (———): simulation results, (- -): $\Delta t = 5T_0$, ($\cdot\ \cdot\ \cdot$):$\Delta t = T_0$, ($-\cdot-\cdot$):$\Delta t = 0.2T_0$.

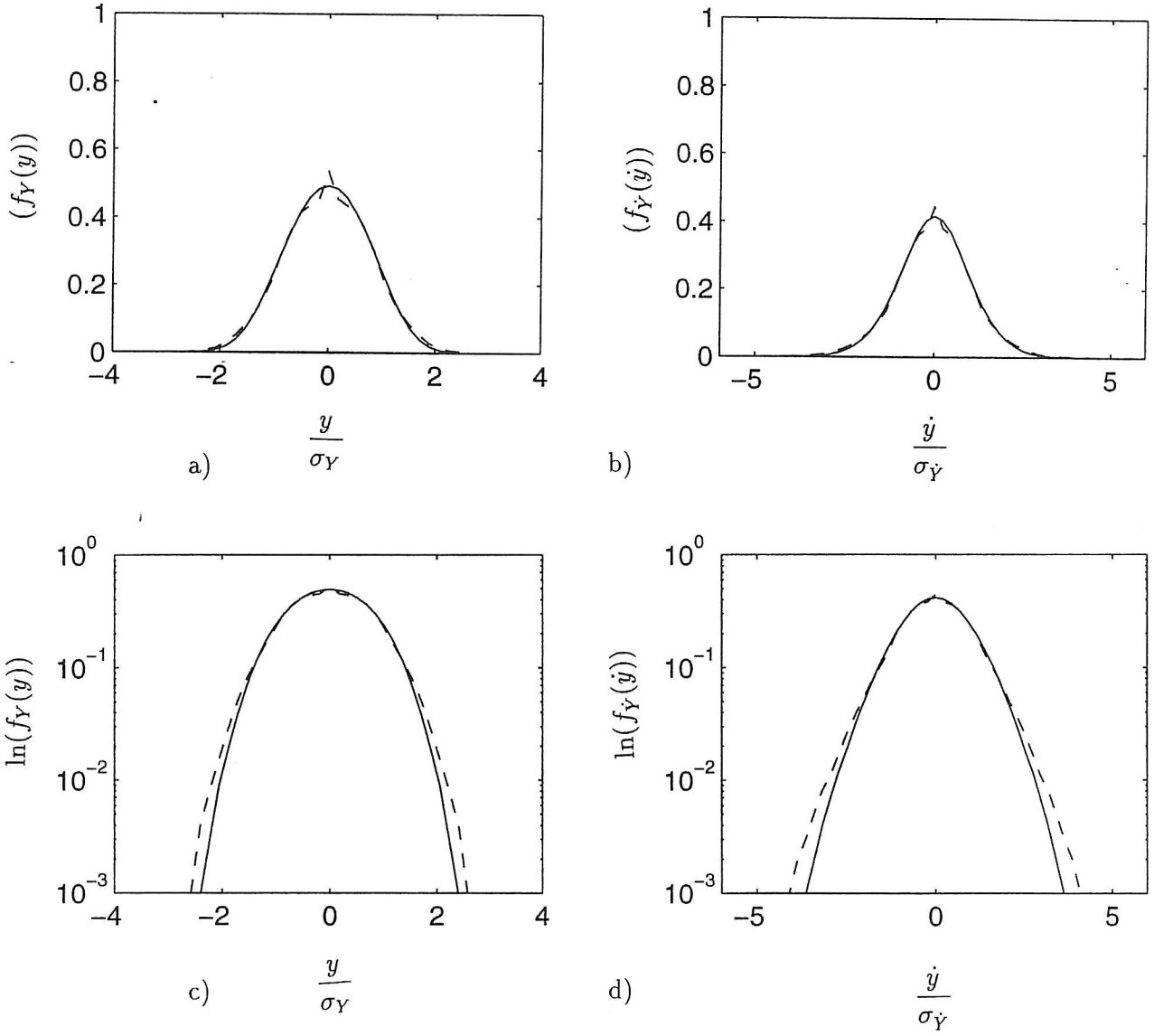

Fig. 4. Case $\nu = 0.2$. Stationary marginal probability densities of the response of Duffing oscillator driven by the renewal impulse process. a) Stationary probability density function $f_Y(y)$ of the displacement response. b) Stationary probability density function $f_{\dot{Y}}(\dot{y})$ of the velocity response. c) Stationary probability density function of the displacement response, logarithmic scale. d) Stationary probability density function of the velocity response, logarithmic scale. (- -): cell-to-cell mapping results, (———): simulation results, $\Delta t = 0.2T_0$.

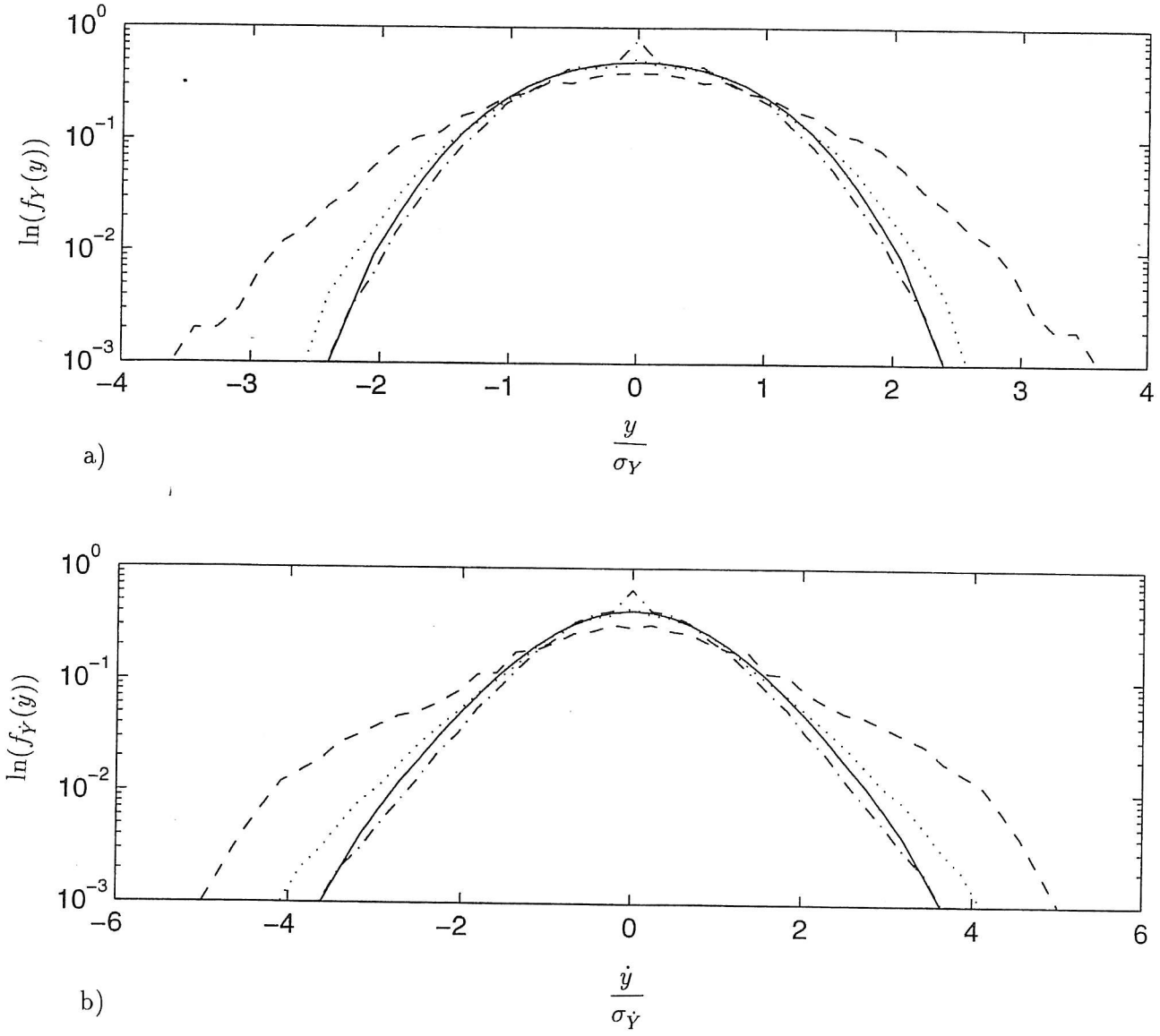

Fig. 5. Case $\nu = 0.2$. Results obtained for different choices of the length Δt of a transition time interval. a) Stationary probability density function of the displacement response, in logarithmic scale. b) Stationary probability density function of the velocity response, in logarithmic scale. (———): simulation results, (- -): $\Delta t = T_0$, (· · · ·): $\Delta t = 0.2T_0$, (—·—·): $\Delta t = 0.1T_0$.

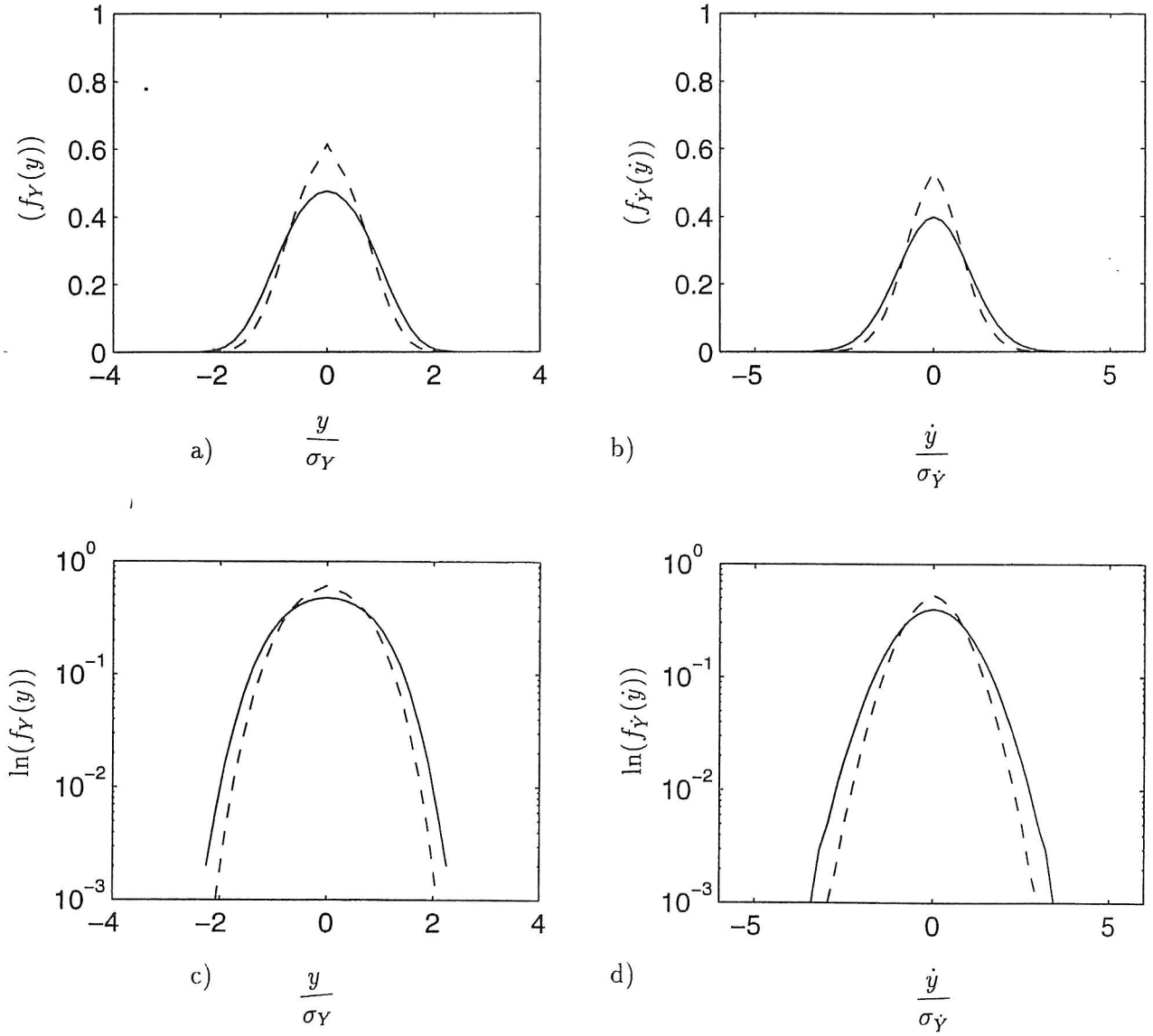

Fig. 6. Case $\nu = 2.0$. Stationary marginal probability densities of the response of Duffing oscillator driven by the renewal impulse process. a) Stationary probability density function $f_Y(y)$ of the displacement response. b) Stationary probability density function $f_{\dot{Y}}(\dot{y})$ of the velocity response. c)Stationary probability density function of the displacement response, logarithmic scale. d) Stationary probability density function of the velocity response, logarithmic scale. (- -): cell-to-cell mapping results, (———): simulation results, $\Delta t = 0.05T_0$.

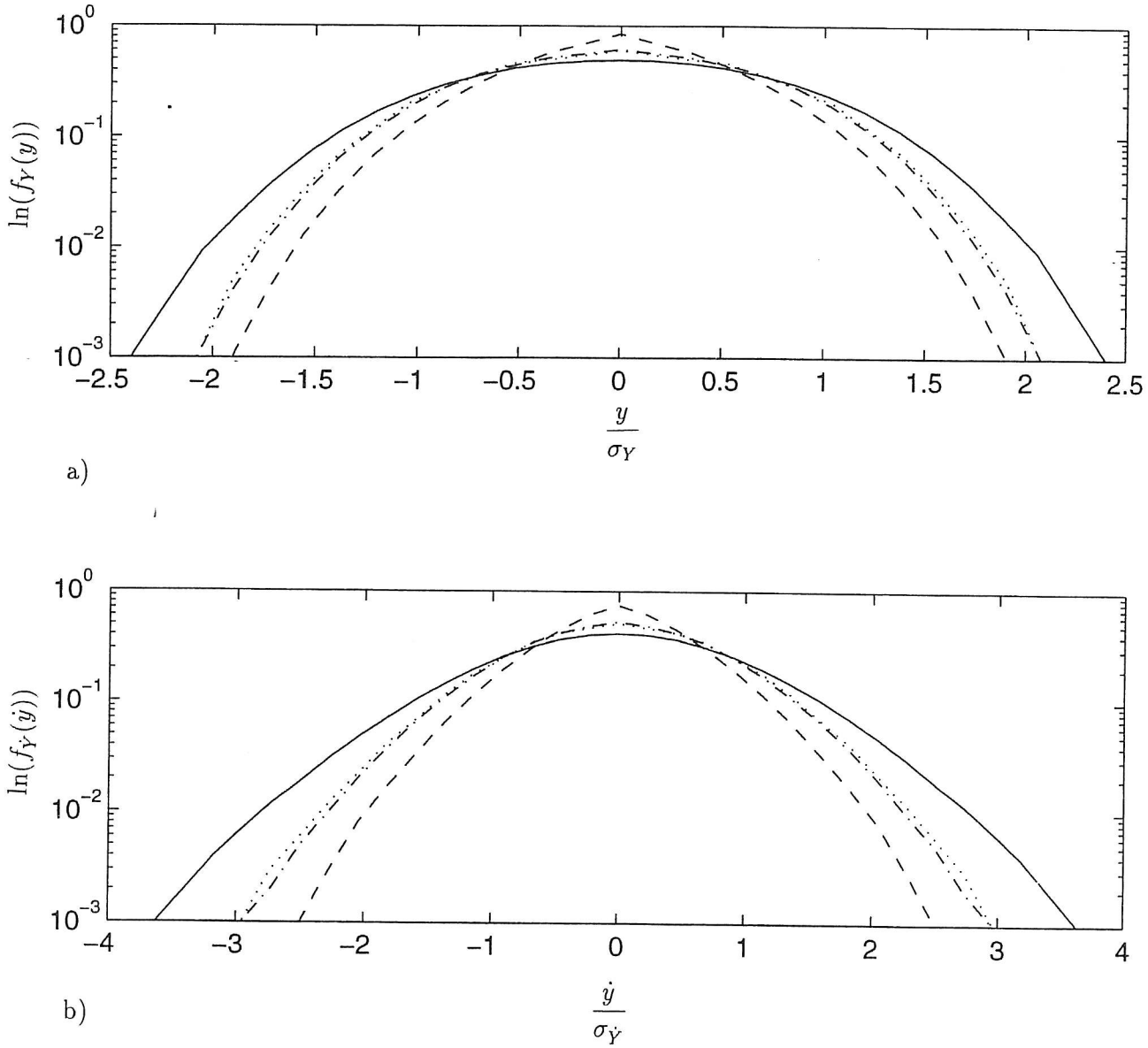

Fig. 7. Case $\nu = 2.0$. Results obtained for different choices of the length Δt of a transition time interval. a) Stationary probability density function of the displacement response, in logarithmic scale. b) Stationary probability density function of the velocity response, in logarithmic scale. (———): simulation results, (- -): $\Delta t = 0.02T_0$, (· · · ·): $\Delta t = 0.1T_0$, (—·—·): $\Delta t = 0.05T_0$

of impulses have been taken into account. Comparison of the obtained stationary response probability densities with those from extensive Monte Carlo simulations shows the validity and high accuracy of the cell-to-cell mapping technique developed.

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